A DUALITY PROOF OF A THEOREM OF P. HILL

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Abstract. A duality argument is given to prove the equivalence of a recent theorem of P. Hill and the main step in Zippin's proof of Ulm's theorem.

Let G and H be isomorphic finite abelian p-groups with isomorphic subgroups A and B, respectively, and set $X = G/A$, $Y = H/B$. In his 1935 proof of Ulm's theorem, Zippin established the following result.

Theorem (Zippin [2, p. 30], [3]). *If $\varphi : A \to B$ is an isomorphism, then there exists an isomorphism $G \to H$ that induces φ if and only if φ preserves heights.*

Recently, P. Hill has proved the following theorem (stated in a slightly different but equivalent form).

Theorem (P. Hill [1]). *If $\psi : X \to Y$ is an isomorphism, then there exists an isomorphism $G \to H$ that induces ψ if and only if ψ preserves coset covalues.*

Hill's theorem appears to be dual to Zippin's theorem. The purpose of this note is to make the connection between these two results more explicit by giving a duality proof of their equivalence.

First, recall the definitions. Let P be any finite abelian p-group and let $0 \neq x \in P$. The *height* of x in P is the unique integer $k \geq 0$ for which $x \in p^k P \backslash p^{k+1} P$ (and the height of 0 is ∞, by convention). Similarly, the *exponent* of x in P is the unique integer $k > 0$ for which $x \in P[p^k] \backslash P[p^{k-1}]$ where $P[p^k] := \{y \in P \mid p^k y = 0\}$ (and the exponent of 0 is zero). If $S \leq P$, then the *coset covalue* of $x + S$ is by definition the least among the exponents of the elements of $x + S$.

With the notation as above, set $A_k = A \cap p^k G$ and $X^k = G/(A + G[p^k])$ and define B_k and Y^k similarly. Let $\pi_X : G \to X$ and $\pi_Y : G \to Y$ denote the canonical epimorphisms.

Lemma.

(i) *An isomorphism $A \to B$ preserves heights if and only if it induces an isomorphism $A_k \to B_k$ for each k.*

(ii) *An isomorphism $X \to Y$ preserves coset covalues if and only if it induces an isomorphism $X^k \to Y^k$ for each k.*

Proof. The proof of (i) is trivial.

(ii) We can make the identifications $X^k = X/\pi_X(G[p^k])$ and $Y^k = Y/\pi_Y(H[p^k])$ so that an isomorphism $X \to Y$ induces an isomorphism $X^k \to Y^k$ if and only if it induces an isomorphism $\pi_X(G[p^k]) \to \pi_Y(H[p^k])$. The statement now follows since clearly the coset covalue of a coset $g + A$ is k if and only if $g + A \in \pi_X(G[p^k]) \backslash \pi_X(G[p^{k-1}])$. □

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In view of this lemma, the equivalence of Zippin's theorem and Hill's theorem can be stated as follows.

Theorem. *The following statements are equivalent.*

(Z) *If $\varphi : A \to B$ is an isomorphism, then there exists an isomorphism $G \to H$ that induces φ if and only if φ induces an isomorphism $A_k \to B_k$ for each k,*
(H) *If $\psi : X \to Y$ is an isomorphism, then there exists an isomorphism $G \to H$ that induces ψ if and only if ψ induces an isomorphism $X^k \to Y^k$ for each k.*

We need some preliminaries before proving the theorem. Denote the contravariant functor $\mathrm{Hom}(\,\cdot\,, \mathbb{Z}(p^\infty))$ on finite abelian p-groups by $P \mapsto P^*$ (object map) and $\varphi \mapsto \varphi^*$ (morphism map). This functor is exact and P^{**} is naturally isomorphic to P (cf. [2, pp. 79–80]). As usual, if $S \le P$ we identify S^* with P^*/S^0 and $(P/S)^*$ with S^0 where $S^0 = \{f \in P^* \mid f(S) = 0\}$.

Lemma 1. *Let $\eta : G \to H$ be a homomorphism.*

(i) *If η induces a homomorphism $\psi : X \to Y$, then η^* induces ψ^*.*
(ii) *If η induces a homomorphism $\varphi : A \to B$, then η^* induces φ^*.*

Proof. (i) Assume η induces a homomorphism $\psi : X \to Y$. This means that $\psi \circ \pi_X = \pi_Y \circ \eta$. Applying $*$, we have $\pi_X^* \circ \psi^* = \eta^* \circ \pi_Y^*$. Since $\pi_X^* : X^* \to G^*$ and $\pi_Y^* : Y^* \to H^*$ are the inclusion maps, this equation says that η^* induces ψ^*.

(ii) Similar. □

Lemma 2.

(i) $A_k{}^* = A^*{}^k$.
(ii) $X^k{}^* = X^*{}_k$.

Proof. (i) $A_k{}^* = G^*/A_k{}^0 = G^*/(A^0 + (p^kG)^0) = A^*{}^k$, since $(p^kG)^0 = G^*[p^k]$.

(ii) Use (i) with $X^* \le G^*$ in place of $A \le G$ and then apply $*$. □

Proof of Theorem. (Z)⇒(H). Assume that statement (Z) holds. One implication in (H) is trivial. For the other, suppose we are given an isomorphism $\psi : X \to Y$ that induces an isomorphism $X^k \to Y^k$ for each k. By Lemma 1(i) and Lemma 2(ii), the isomorphism $\psi^* : Y^* \to X^*$ induces an isomorphism $Y^*{}_k \to X^*{}_k$ for each k. Part (Z) of the theorem applies to give an isomorphism $\eta : H^* \to G^*$ that induces ψ^*. Using Lemma 1(ii) and natural isomorphisms, we obtain an isomorphism $G \to H$ that induces ψ.

(H)⇒(Z). Similar. □

References

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